XXII. *An improved Solution of a Problem in physical Astronomy; by which, swiftly converging Series are obtained, which are useful in computing the Perturbations of the Motions of the Earth, Mars, and Venus, by their mutual Attraction. To which is added an Appendix, containing an easy Method of obtaining the Sums of many slowly converging Series which arise in taking the Fluents of binomial Surds, &c. By the Rev.* John Hellins, *F. R. S. Vicar of Potter's Pury, in Northamptonshire. In a Letter to the Rev.* Nevil Maskelyne, *D. D. F. R. S. and Astronomer Royal.*

Read June 28, 1798.

REVEREND SIR, Potter's Pury, April 17, 1797.

SUCH is the subject of the inclosed paper, and such the reputation for skill and industry, which the many valuable papers you have communicated to the Royal Society, and your other learned works, have justly procured to you, that it could not with more propriety be submitted to the judgment of any other person than yourself, even if the writer of it were a stranger to you.

But there are circumstances which render my presenting it to you, in some measure, a duty. I had the advantage of being, for some years, your Assistant in the Royal Observatory at Greenwich; during which time, you made the important observations on the mountain *Schehallien*, in Scotland, which afford

an ocular demonstration of the attraction of that mountain, and a strong argument for the general attraction of matter, a subject nearly connected with that of the following pages; and it was from you that I received the problem of which you will here find an improved solution.

The diffidence with which I entered on a speculation which had engaged the attention of such learned men as SIMPSON, EULER, and DE LA GRANGE, is well known to you. Considering the great abilities of these men, and the length of time which EULER, in particular, appears to have employed on the subject, all that I at first expected to effect was, to facilitate the summation of the slowly converging series by means of which they had computed the perturbations of the motions of the planets in their orbits, which arise from their actions on one another, by the force of gravity; and that this might be done by a method which I had some time before discovered, was evident, on inspecting their series. Here, it is probable, I should have stopped, had not you been pleased to put into my hands a sheet of paper, written by the late Mr. SIMPSON, which, though very ingenious, was, by mistakes, which seem to have entered in transcribing it, rendered unintelligible to some eminent mathematicians who had perused it; in which state it had remained thirty-six years. On perusing this paper, the first thing that occurred to me was, a different method of finding the fluent, from that which had been used by Mr. SIMPSON; by which means, series converging by the powers of $\frac{1}{6}$ were obtained, while the series brought out the common way lost all convergency by a geometrical progression, and a computation by it was more difficult than the computation of the length of

a quadrantal arch of the circle by the series $1 + \frac{1}{2 \cdot 3} + \frac{3}{2 \cdot 4 \cdot 5} + \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7}$, *&c.* Afterwards, I discovered the method of transforming that series which had lost all convergency by a geometrical progression, into another in which the literal powers decrease very swiftly; which is the improvement I now offer to you.

In comparing the series here produced, for computing the values of A and B in the equation $(a - b \times \cos. z)^{-n} = A + B'. \cos. z + C . \cos. 2z + D . \cos. 3z +$ *&c.* with those which have been published for that purpose, by Messrs. EULER and DE LA GRANGE, it will appear, that those cases which were the most difficult to be computed by their methods, are the most easy by mine. For instance, if Venus's perturbation of the motion of the Earth were to be computed, (and *vice versâ,*) the literal powers which have place in M. EULER's series, would be very nearly equal to the powers of $\frac{9}{10}$; the literal powers which have place in M. DE LA GRANGE's series, would be nearly equal to the powers of $\frac{1}{2}$; and, in the series now produced, the literal powers would decrease somewhat swifter than the powers of $\frac{1}{36}$.

M. DE LA GRANGE has indeed, by a very ingenious device, obtained a convergency in the numeral coefficients of the series that he uses, which, for the first five terms of it, is nearly equal to the powers of $\frac{1}{4}$; but this convergency becomes less and less in every succeeding term, and the coefficients approach pretty fast to a ratio of equality; so that, to obtain the sum of the series to six places of decimals, he proposes to compute the first ten terms of it. The case in which those coefficients

have that convergency, is when n (which answers to his s,) is $= \frac{-1}{2}$, a case which does not often happen; however, from the values of A and B, when $n = \frac{-1}{2}$, he derives their values when $n = \frac{1}{2}, \frac{3}{2}$, &c. by another very ingenious device, worthy of that skill for which he is justly celebrated. But, by the method now proposed, the chief part of the convergency is in the literal powers; and such a difference in the numeral coefficients, for a different value of n, does not take place.

For Mars's perturbation of the Earth's motion, the literal powers by which the three different series converge, are nearly as follows:

M. EULER's,	by the powers of	$\frac{4}{5}$;
M. DE-LA GRANGE's,		$\frac{10}{23}$;
The series now proposed,		$\frac{1}{20}$.*

If, indeed, the perturbation which arises from the action of Jupiter upon the earth was to be computed, M. DE LA GRANGE's series would be the best that has hitherto been published for the purpose, as the literal powers of it would, in that case, be

* For obtaining nearly the different rates of convergency of the literal powers in the three series, it will be sufficient to consider the distance of the two planets of which the perturbations are to be computed, as $= \sqrt{(RR + rr - 2Rr \times c, z)}$, where R and r denote their mean distances from the sun, of which R is the greater, and c, z the cosine of the angle of commutation. Then will M. DE LA GRANGE's series converge by the powers of the quantity $\frac{rr}{RR}$; and, since $RR + rr = a$, and $2Rr = b$, in our notation, and the converging quantity in M. EULER's series is $(nn) = \frac{bb}{aa}$, it will be $= \frac{4R^2r^2}{(RR+rr)^2}$; and cc, by the powers of which the new series converges, is $= \frac{a-b}{a+b} = \frac{RR - 2Rr + rr}{RR + 2Rr + rr} = \frac{(R-r)^2}{(R+r)^2}$. See the *Memoirs of the Royal Academy of Sciences and Belles-Lettres* at Berlin, for 1781, p. 257; M. EULER's *Institutiones Calculi Integralis*, Vol. I. p. 186; and Art. 4, in what follows.

nearly equal to the powers of $\frac{1}{27}$, while the literal powers in the new series would differ but little from those of $\frac{11}{24}$. So that, for computing the perturbation of each of these three planets, we now have series converging so very swiftly, that the first four terms are sufficient for the purpose.

These indeed are the perturbations of motion, arising from the actions of the planets, which the inhabitants of this globe have most frequent occasion to compute. And, since two of the three are most easily calculated by the method explained in the following pages, I am not without hopes that I have rendered an acceptable piece of service to astronomers in general, and more especially to those who are most intent upon improving astronomical tables.

But it may be proper to remark, that the use of the new series is not confined to the computations just mentioned, but may successfully be used in computing the perturbations of the motions of other planets. For instance, in the computation of the perturbation of Saturn's motion by Jupiter, (and *vice versâ,*) the convergency of this series will be nearly by the powers of $\frac{1}{12}$, which is a swift rate of convergency. And, for the perturbation of the *Georgium sidus* by Saturn, (and *vice versâ,*) the series will converge nearly by the powers of $\frac{1}{9}$, which is also swiftly.

And it is further to be remarked, that in the last instance, and indeed whenever the radii of the orbits of the two planets differ from each other in the ratio of 2 to 1, M. De la Grange's series may be used with advantage, since the convergency of the first five terms of it will then be nearly by the powers of $\frac{1}{16}$; the numeral coefficients of those terms converging as

swiftly as the literal powers do in that case. And, when the ratio of the two radii is greater than that of 2 to 1, his series will converge more swiftly.

With great pleasure therefore I see, that, by one or other of these methods, some of the longest and most difficult calculations which formerly arose in the theory of astronomy, may now be exchanged for others which are short and easy.

It is with satisfaction also, that I perceive the facility of computing by the series I now present to you, is not at all lessened by the more general notation you have given to the denominator of the fraction from which it is derived, at the same time that a more accurate result is obtained than M. DE LA GRANGE proposed. For, in the computations of which I have been speaking, he neglected both the excentricities of the orbits of the planets, and their inclinations to the ecliptic, as inconsiderable: you, finding the effect of these omissions to be greater than he imagined, have taken them in. Your other ingenious labours on this subject will be best described by yourself, and cannot fail of being gratefully received by all learned astronomers.

With respect to the method by which the sums of the very slowly converging numerical series, which occur in the subsequent pages, are obtained, I need not say to you, that it is of extensive utility, and may be successfully applied in many cases.

I have only to request, that, if the paper here inclosed meets with your approbation, you will communicate it to the Royal Society. For, although I think I cannot be mistaken re-

specting the utility of the invention explained in it, yet such is my respect for that learned body, that I am unwilling to send them any paper of mine, on so difficult and important a subject, till it has been examined by an able judge of the subject.

I am,

Rev. Sir, *&c.*

JOHN HELLINS.

An improved Solution of a Problem in physical Astronomy, &c.

I

1. The perturbation of the motions of the planets in their orbits, by the action of one upon another, is a curious phænomenon, which, while it affords to the philosopher a clear proof of the general attraction of matter, produces a problem of no small difficulty to the astronomer; *viz.* to compute the quantity by which a planet, so acted upon, deviates from an ellipsis in its course round the sun: a problem which hath called forth the skill of several of the most learned philosophers and astronomers of the last and present age.

A preparatory step to the solution of this problem is, to find a convenient expression for the reciprocal of the cube, or rather of the n^{th} power, of the distance of any two planets. Such an expression was first given by M. EULER, in series proceeding by the cosines of the multiples, in arithmetic progression, of the angle of commutation; but the calculations of the first

two coefficients in it were very laborious, requiring the summation of series of the common form, which converged very slowly. Afterwards, other series were discovered by other authors, whereby the same coefficients might be computed with less labour; the best of which, that I have seen, appear to be those that were pointed out to me by Dr. MASKELYNE, invented by M. DE LA GRANGE, and published in the *Memoirs of the Royal Academy of Sciences at Berlin*, for the year 1781. Yet, the calculation of the two first coefficients, A and B, for the perturbations of Mars, Venus, and the Earth, by his method, is not shorter, if it be so short as by my method, to the investigation of which I now proceed.

PROBLEM.

2. *To determine the values of* A, B, C, D, &c. *in the equation* $\frac{\dot{z}}{(a-b\,.\,\cos.\,z)^n} = \dot{z}\,(A+B\,.\,\cos.\,z+C\,.\,\cos.\,2z+D\,.\,\cos.\,3z,\,\&c.)$ z *being the arch of a circle of which the radius is* 1, *and* b *less than* a.

First, *to find the coefficient* A.

3. The fluent of the right-hand side of this equation is $Az + B\,.\,\sin.\,z + \frac{1}{2}C\,.\,\sin.\,2z + \frac{1}{3}D\,.\,\sin.\,3z + \frac{1}{4}E\,.\,\sin.\,4z$,* &c. which evidently vanishes when $z=0$; and, when $z=3\cdot14159$, &c. the arch of 180°, it becomes barely $=Az$, the sines of z, $2z$, $3z$, &c. being then each $=0$. If, therefore, the fluent of the first side of the equation be taken, the increase of it, while z increases from 0 to $3\cdot14159$ &c. $=\pi$, will be $=\pi A$; and, consequently, A will be determined.

* See M. EULER's *Institutiones Calculi Integralis*, Vol. I. p. 150.

4. Now, to find the fluent of $\frac{\dot{z}}{(a-b\,.\,\cos.\,z)^n}$, we have $\frac{\dot{z}}{(a-b\,.\,\cos.\,z)^n} = \frac{-\dot{x}}{\sqrt{(1-xx)}\,(a-bx)^n}$, x being put $=$ the cosine of z; in which expressions, while z increases from 0 to $3\cdot14159$, x will decrease from 1 to -1. Therefore, to obtain a more convenient expression, put $vv = \frac{a-bx}{a+b}$; then, while x decreases from 1 to -1, vv will increase from $\frac{a-b}{a+b}$ to $\frac{a+b}{a+b} = 1$; and we shall have the following equations:

$$a-bx = (a+b)\,vv,\ (a-bx)^n = (a+b)^n v^{2n},\ x = \frac{a-(a+b)vv}{b},\ -\dot{x} = \frac{(a+b)2v\dot{v}}{b};\ 1+x = 1+\frac{a-(a+b)vv}{b} = \frac{a+b-(a+b)vv}{b} = \frac{a+b}{b}(1-vv)$$

$$1-x = 1-\frac{a-(a+b)vv}{b} = \frac{b-a+(a+b)vv}{b} = \frac{a+b}{b}\left(\frac{b-a}{a+b}+vv\right) = \frac{a+b}{b}\left(vv-\frac{a-b}{a+b}\right) = \frac{a+b}{b}(vv-cc),$$

cc being put $= \frac{a-b}{a+b}$. And from these equations, the three following are easily obtained, viz.

$$\sqrt{(1+x)}\,\sqrt{(1-x)} = \sqrt{\left(\frac{a+b}{b}(1-vv)\right)} \times \sqrt{\left(\frac{a+b}{b}(vv-cc)\right)} = \frac{a+b}{b}\sqrt{\left((1-vv)\,(vv-cc)\right)};\ \text{and,}$$

$$\frac{-\dot{x}}{\sqrt{(1-xx)}} = \frac{(a+b)2v\dot{v}}{b} \times \frac{b}{(a+b)\sqrt{\left((1-vv)\,(vv-cc)\right)}} = \frac{2v\dot{v}}{\sqrt{(1-vv)}\sqrt{(vv-cc)}};$$

and, lastly, $\frac{-\dot{x}}{\sqrt{(1-xx)}\,(a-bx)^n} = \frac{2v\dot{v}}{\sqrt{(1-vv)}\sqrt{(vv-cc)}\,(a+b)^n v^{2n}} = (a+b)^{-n} \times \frac{2\dot{v}v^{1-2n}}{\sqrt{(1-vv)}\sqrt{(vv-cc)}}$; the fluent of which may be found when the value of n is given.

5. Now, the values of n with which astronomers are most concerned, are $\frac{3}{2}$ and $\frac{5}{2}$. Let, therefore, $\frac{3}{2}$ be written for n, and the radical quantity $\sqrt{(1-vv)}$ be converted into series, and the last expression will be

$$=(a+b)^{-\frac{3}{2}}\times\frac{2\dot{v}v^{-2}}{\sqrt{(vv-cc)}}\left(1+\frac{vv}{2}+\frac{3v^4}{2.4}+\frac{3.5v^6}{2.4.6}+\frac{3.5.7v^8}{2.4.6.8},\ \&c.\right)$$

$$=(a+b)^{-\frac{3}{2}}\times\left\{\begin{array}{l}\dfrac{2\dot{v}v^{-2}}{\sqrt{(vv-cc)}}\\ +\dfrac{\dot{v}}{\sqrt{(vv-cc)}}\left(1+\dfrac{3vv}{4}+\dfrac{3.5v^4}{4.6}+\dfrac{3.5.7v^6}{4.6.8},\ \&c.\right)\end{array}\right.$$

And the fluents of these several terms, without their coefficients, are as follows:

$$\int\frac{\dot{v}v^{-2}}{\sqrt{(vv-cc)}}\ \text{is}=\frac{\sqrt{(vv-cc)}}{ccv};$$

$$\int\frac{\dot{v}}{\sqrt{(vv-cc)}}=\text{H.L.}\frac{v+\sqrt{(vv-cc)}}{c}=\alpha;$$

$$\int\frac{\dot{v}v^2}{\sqrt{(vv-cc)}}=\frac{\sqrt{(vv-cc)}\,v+cc\alpha}{2}=\beta;$$

$$\int\frac{\dot{v}v^4}{\sqrt{(vv-cc)}}=\frac{\sqrt{(vv-cc)}\,v^3+3cc\beta}{4}=\gamma;$$

$$\int\frac{\dot{v}v^6}{\sqrt{(vv-cc)}}=\frac{\sqrt{(vv-cc)}\,v^5+5cc\gamma}{6}=\delta;$$

&c. &c.

These fluents, being multiplied by their proper coefficients, and collected together, and the whole multiplied by the common factor $(a+b)^{-\frac{3}{2}}$, the fluent sought will be

$$(a+b)^{-\frac{3}{2}}\times\left\{\begin{array}{l}\dfrac{2\sqrt{(vv-cc)}}{ccv}\\ +\alpha+\dfrac{3}{4}\beta+\dfrac{3.5}{4.6}\gamma+\dfrac{3.5.7}{4.6.8}\delta,\ \&c.\end{array}\right.$$

6. We must now inquire what value this series has when $z=0$; in which case, x being $=1$, vv is $=\frac{a-b}{a+b}=cc$. And it will appear that, with this value of vv, every term of the series vanishes, so that the fluent needs no correction. If, therefore, we compute the value of this series when $z=\pi$, *i. e.* when $x=-1$, and $vv=\frac{a+b}{a+b}=1$, we shall have the value of $A\pi$,

and consequently, A will be determined. But, with this value of v, the terms β, γ, δ, *&c.* lose all convergency by the geometrical progression v, v^3, v^5, *&c.* and the computation of the value of the series, by the common method, would be more laborious than the computation of the quadrantal arch of the circle, by the series $1 + \frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7}$, *&c.* Here then we are stopped. But, by contemplating this series, expressed in terms of α and c, as it stands below, a very different method of obtaining the value of it is suggested.

$$
\begin{array}{llllllll}
7. & \alpha = & \alpha & & & & & \\
& \frac{3}{4}\beta = & \frac{3\sqrt{(1-cc)}}{4.2} & + & \frac{3cc\alpha}{4.2} & & & \\
& \frac{3.5}{4.6}\gamma = & \frac{3.5\sqrt{(1-cc)}}{4.6.4} & + & \frac{3.5.3cc\sqrt{(1-cc)}}{4.6.4.2} & + & \frac{3.5.3c^4\alpha}{4.6.4.2} & \\
& \frac{3.5.7}{4.6.8}\delta = & \frac{3.5.7\sqrt{(1-cc)}}{4.6.8.6} & + & \frac{3.5.7.5cc\sqrt{(1-cc)}}{4.6.8.6.4} & + & \frac{3.5.7.5.3c^4\sqrt{(1-cc)}}{4.6.8.6.4.2}, & \&c. \\
& \frac{3.5.7.9}{4.6.8.10}\varepsilon = & \frac{3.5.7.9\sqrt{(1-cc)}}{4.6.8.10.8} & + & \frac{3.5.7.9.7cc\sqrt{(1-cc)}}{4.6.8.10.8.6} & + & \frac{3.5.7.9.7.5c^4\sqrt{(1-cc)}}{4.6.8.10.8.6.4}, & \&c. \\
& \&c. & \&c. & & \&c. & & \&c. &
\end{array}
$$

Here it appears, 1st. That the geometrical progression 1, cc, c^4, *&c.* has place in the first, second, third, *&c.* columns of quantities on the right-hand side of the equation, the terms of which, when b is nearly $= a$, decrease very swiftly.

2dly. That, in the diagonal line of quantities in which α enters, besides this decrease of the terms, by the literal powers before mentioned, the numeral coefficients are so simple that a considerable number of the terms may quickly be computed.

3dly. That, if this diagonal line of quantities be taken away, the first, second, third, *&c.* infinite columns of quantities which

remain will have the literal factors $\surd(1-cc)$, $cc\surd(1-cc)$, $c^4\surd(1-cc)$, &c. respectively, which are in the progression before mentioned.

4thly. That, if the sum of the infinite series of numeral coefficients below the line, in each of these columns, can be obtained, then the original series, which had lost all convergency by the literal powers v, v^3, v^5, &c. may be transformed into two others, in which the literal powers will be cc, c^4, &c.

8. But the sums of these infinite series are attainable, and are as follows:

$$\frac{3}{4.2} + \frac{3.5}{4.6.4} + \frac{3.5.7}{4.6.8.6} + \frac{3.5.7.9}{4.6.8.10.8}, \text{ \&c. is } = \tfrac{1}{2} + \text{H. L. } 2 = \lambda;$$

$$\frac{3.5.3}{4.6.4.2} + \frac{3.5.7.5}{4.6.8.6.4} + \frac{3.5.7.9.7}{4.6.8.10.8.6} + \frac{3.5.7.9.11.9}{4.6.8.10.12.10.8}, \text{ \&c. is } = \tfrac{11}{32} + \tfrac{7}{8}\text{H. L. } 2 = \mu;$$

$$\frac{3.5.7.5.3}{4.6.8.6.4.2} + \frac{3.5.7.9.7.5}{4.6.8.10.8.6.4} + \frac{3.5.7.9.11.9.7}{4.6.8.10.12.10.8.6}, \text{ \&c. is } = \frac{17 + 51\text{ H. L. } 2}{64} = \nu;^*$$

&c. &c.

But these three sums are as many as are requisite, when the perturbation of the motion of either the Earth, Mars, or Venus, by the attraction of any one of the other, is to be computed.

9. The sum of the coefficients in the first, second, and third columns in Art. 7. being now obtained, take $\frac{8-2cc}{8-5cc}\alpha$ for the value of the series $\alpha\left(1 + \frac{3cc}{4.2} + \frac{3.5.3c^4}{4.6.4.2}, \text{ \&c.}\right)$, which will be exact enough for the purpose, and we shall have, by Art. 3, 5, 6, 7, and 8,

* See the Appendix.

$$\pi A=(a+b)^{-\frac{3}{2}}\times\begin{cases}\dfrac{2\sqrt{(1-cc)}}{cc}\\+\dfrac{8-2cc}{8-5cc}\alpha\\+\lambda\sqrt{(1-cc)}+\mu cc\sqrt{(1-cc)}+\nu c^4\sqrt{(1-cc)};\end{cases}$$

and thence, by a more commodious arrangement of the terms, and dividing both sides by π,

$$A=\frac{1}{\pi(a+b)^{\frac{3}{2}}}\times\begin{cases}\dfrac{8-2cc}{8-5cc}\alpha\\+\sqrt{(1-cc)}\left(\dfrac{2}{cc}+\lambda+\mu cc+\nu c^4\right)\end{cases}$$

10. The value of A, when $n=\frac{3}{2}$, being now found, let us next investigate the value of it when $n=\frac{5}{2}$; which, for the sake of distinction, in a use to be made of it in a subsequent article, I denote by A'.

By writing $\frac{5}{2}$ for n in the fluxionary expression obtained in Art. 4. we have $(a+b)^{-\frac{5}{2}}\times\frac{2\dot{v}v^{-4}}{\sqrt{(1-vv)}\sqrt{(vv-cc)}}$, which, by converting the radical quantity $\sqrt{(1-vv)}$ into series, becomes

$$(a+b)^{-\frac{5}{2}}\times\frac{2\dot{v}v^{-4}}{\sqrt{(vv-cc)}}\left(1+\frac{vv}{2}+\frac{3v^4}{2.4}+\frac{3.5v^6}{2.4.6}+\frac{3.5.7v^8}{2.4.6.8},\ \&c.\right)$$

$$=(a+b)^{-\frac{5}{2}}\times\begin{cases}\dfrac{2\dot{v}v^{-4}}{\sqrt{(vv-cc)}}+\dfrac{\dot{v}v^{-2}}{\sqrt{(vv-cc)}}\\+\dfrac{\dot{v}}{\sqrt{(vv-cc)}}\left(\dfrac{3}{4}+\dfrac{3.5vv}{4.6}+\dfrac{3.5.7v^4}{4.6.8}+\dfrac{3.5.7.9v^6}{4.6.8.10},\ \&c.\right)\end{cases}$$

Now, the fluents of these terms, without their coefficients, are as follows:

$$\int\frac{\dot{v}v^{-4}}{\sqrt{(vv-cc)}}\ \text{is}=\frac{\sqrt{(vv-cc)}}{3ccv^3}+\frac{2\sqrt{(vv-cc)}}{3c^4v};$$

$$\int\frac{\dot{v}v^{-2}}{\sqrt{(vv-cc)}}=\frac{\sqrt{(vv-cc)}}{ccv};$$

$$\int\frac{\dot{v}}{\sqrt{(vv-cc)}}=\text{H. L.}\frac{v+\sqrt{(vv-cc)}}{c}=\alpha;$$

$$\int\frac{\dot{v}v^2}{\sqrt{(vv-cc)}}=\frac{\sqrt{(vv-cc)}v+cc\alpha}{2}=\beta;$$ and the rest as they are exhibited in Art. 5.

These fluents being multiplied by their proper coefficients, and collected together, and their sum multiplied by the common factor $(a+b)^{-\frac{5}{2}}$, we shall have

$$(a+b)^{-\frac{5}{2}} \times \left\{ \begin{array}{l} \frac{2\sqrt{(vv-cc)}}{3ccv^3} + \frac{4\sqrt{(vv-cc)}}{3c^4v} + \frac{\sqrt{(vv-cc)}}{ccv} \\ + \frac{3}{4}\alpha + \frac{3.5}{4.6}\beta + \frac{3.5.7}{4.6.8}\gamma + \frac{3.5.7.9}{4.6.8.10}\delta, \text{ \&c.} \end{array} \right.$$

which fluent needs no correction, since, when $v=c$, the whole vanishes. The value of it therefore, when $v=1$, will be $=A'\pi$, which is what we want; and, in obtaining it, the only difficulty is, to compute the series in which α, β, γ, &c. enter, which may be overcome in the manner shewn above in Art. 7. For the value of this series, when $v=1$, will be as follows:

11. $$\frac{3}{4}\alpha = \frac{3}{4}\alpha$$

$$\frac{3.5}{4.6}\beta = \frac{3.5\sqrt{(1-cc)}}{4.6.2} + \frac{3.5cc\alpha}{4.6.2}$$

$$\frac{3.5.7}{4.6.8}\gamma = \frac{3.5.7\sqrt{(1-cc)}}{4.6.8.4} + \frac{3.5.7.3cc\sqrt{(1-cc)}}{4.6.8.4.2} + \frac{3.5.7.3c^4\alpha}{4.6.8.4.2}$$

$$\frac{3.5.7.9}{4.6.8.10}\delta = \frac{3.5.7.9\sqrt{(1-cc)}}{4.6.8.10.6} + \frac{3.5.7.9.5cc\sqrt{(1-cc)}}{4.6.8.10.6.4} + \frac{3.5.7.9.5.3c^4\sqrt{(1-cc)}}{4.6.8.10.6.4.2}, \text{ \&c.}$$

$$\frac{3.5.7.9.11}{4.6.8.10.12}\varepsilon = \frac{3.5.7.9.11\sqrt{(1-cc)}}{4.6.8.10.12.8} + \frac{3.5.7.9.11.7cc\sqrt{(1-cc)}}{4.6.8.10.12.8.6} + \frac{3.5.7.9.11.7.5c^4\sqrt{(1-cc)}}{4.6.8.10.12.8.6.4}, \text{ \&c.}$$

&c. &c. &c. &c.

Here, by attending to the same things which were observed in Art. 7, we may easily obtain the values of as many of the infinite columns of quantities on the right-hand side of the equation as are wanted; which, for the planets Mars, Venus, the Earth, and some of the rest, are but three.

12. The value of the series, of which α is the factor, *viz.* $\alpha\left(\frac{3}{4} + \frac{3.5cc}{4.6.2} + \frac{3.5.7.3c^4}{4.6.8.4.2}, \text{ \&c.}\right)$, will be obtained sufficiently near for the purpose, by this expression, $\frac{96-23cc}{128-84cc}\alpha$, which is somewhat more exact than the three first terms of it; and the infinite series

$$\frac{3.5}{4.6.2} + \frac{3.5.7}{4.6.8.4} + \frac{3.5.7.9}{4.6.8.10.6} + \frac{3.5.7.9.11}{4.6.8.10.12.8}, \text{ \&c. is}$$

$$= \frac{9}{16} + \frac{3}{4}\text{H. L. } 2 = \lambda';$$

$$\frac{3.5.7.3}{4.6.8.4.2} + \frac{3.5.7.9.5}{4.6.8.10.6.4} + \frac{3.5.7.9.11.7}{4.6.8.10.12.8.6}, \text{ \&c. is}$$

$$= \frac{79}{192} + \frac{11}{16}\text{H. L. } 2 = \mu';$$

$$\frac{3.5.7.9.5.3}{4.6.8.10.6.4.2} + \frac{3.5.7.9.11.7.5}{4.6.8.10.12.8.6.4} + \frac{3.5.7.9.11.13.9.7}{4.6.8.10.12.14.10.8.6}, \text{ \&c. is}$$

$$= \frac{4067}{12288} + \frac{329}{512}\text{H.L. } 2 = \nu'.^*$$

We therefore now have $\frac{96-23cc}{128-84cc}\alpha + \lambda'\surd(1-cc) + \mu' cc\surd(1-cc) + \nu' c^4 \surd(1-cc)$ for a near value of the infinite series $\frac{3}{4}\alpha + \frac{3.5}{4.6}\beta + \frac{3.5.7}{4.6.8}\gamma$, &c.

13. Having thus obtained a sufficiently near value of the infinite series which entered into the fluent, in Art. 10, we have only to add to it the three radical terms there found, v being put $= 1$, and to multiply the whole by $(a+b)^{-\frac{5}{2}}$, and we shall have

$$\pi A' = (a+b)^{-\frac{5}{2}} \times \begin{cases} \frac{2\surd(1-cc)}{3cc} + \frac{4\surd(1-cc)}{3c^4} + \frac{\surd(1-cc)}{cc} \\ + \frac{96-23cc}{128-84cc}\alpha \\ + \lambda'\surd(1-cc) + \mu' cc\surd(1-cc) + \nu' c^4\surd(1-cc); \end{cases}$$

which equation being more concisely expressed, and divided by π, gives

$$A' = \frac{1}{\pi\,(a+b)^{\frac{5}{2}}} \times \begin{cases} \frac{96-23cc}{128-84cc}\alpha \\ + \surd(1-cc)\left(\frac{4+5cc}{3c^4} + \lambda' + \mu' cc + \nu' c^4\right). \end{cases}$$

* See the Appendix.

Secondly, *to find the Coefficient* B.

14. Multiply the equation in Art. 2. by $2\cos.z = 2x$, and we shall have $\frac{2x\dot{z}}{(a-bx)^n} = \dot{z}(\text{A}\times 2\cos.z + \text{B}\times 2\cos.z\times\cos.z + \text{C}\times 2\cos.z \times \cos.2z + \text{D}\times 2\cos.z\times\cos.3z$, &c.); which, because $2\cos.z \times \cos.mz$ is $= \cos.(m-1)z + \cos.(m+1)z$, will be $= \dot{z}$ $(2\text{A}.\cos.z + \text{B}(1+\cos.2z) + \text{C}(\cos.z+\cos.3z) + \text{D}(\cos.2z + \cos.4z)$, &c.)* And, by taking the fluents, we have $\int\frac{2x\dot{z}}{(a-bx)^n} = 2\text{A}.\sin.z + \text{B}z + \frac{1}{2}\text{B}.\sin.2z + \text{C}(\sin.z + \frac{1}{3}\sin.3z) + \text{D}(\frac{1}{2}\sin.2z + \frac{1}{4}\sin.4z)$, &c.; which equation, when $z = 3{\cdot}14159$, &c. $= \pi$, becomes $\int\frac{2x\dot{z}}{(a-bx)^n}$ = barely $\text{B}z = \text{B}\pi$, the sines of z, $2z$, $3z$, &c. being then $= 0$.

15. Now it appears, by the notation in Art. 4, that $\frac{\dot{z}}{(a-bx)^n} = \frac{-\dot{x}}{\sqrt{(1-xx)}\,(a-bx)^n} = (a+b)^{-n} \times \frac{2\dot{v}v^{1-2n}}{\sqrt{(1-vv)}\sqrt{(vv-cc)}}$, and that $x = \frac{a-(a+b)vv}{b}$; we therefore have, by proper substitution,

$$\frac{2x\dot{z}}{(a-bx)^n} = \frac{-2x\dot{x}}{\sqrt{(1-xx)}\,(a-bx)^n} = \left.\begin{array}{l} \frac{2a}{b(a+b)^n}\times\frac{2\dot{v}v^{1-2n}}{\sqrt{(1-vv)}\sqrt{(vv-cc)}} \\ \frac{-2}{b(a+b)^{n-1}}\times\frac{2\dot{v}v^{3-2n}}{\sqrt{(1-vv)}\sqrt{(vv-cc)}} \end{array}\right\},$$

of which two fluxions the fluents may be found, when n has any particular value.

16. First, let n be $\frac{3}{2}$; then the last expression in the preceding article becomes $\frac{2a}{b(a+b)^{\frac{3}{2}}} \times \frac{2\dot{v}v^{-2}}{\sqrt{(1-vv)}\sqrt{(vv-cc)}} - \frac{2}{b(a+b)^{\frac{1}{2}}} \times \frac{2\dot{v}}{\sqrt{(1-vv)}\sqrt{vv-cc)}}$. Now, the fluent of the affirmative part of this expression is evidently $= \frac{2a}{b} \times$ the fluent of the fluxion in Art. 5,

* See SIMPSON's Miscellaneous Tracts, lemma I. p. 76.

that is, $=\frac{2a}{b}A\pi$; and the negative part, by converting $\sqrt{(1-vv)}$ into series, will become $\frac{-2}{b(a+b)^{\frac{1}{2}}} \times \frac{2\dot{v}}{\sqrt{(vv-cc)}}\left(1+\frac{vv}{2}+\frac{3v^4}{2.4}+\frac{3.5v^6}{2.4.6}, \&c.\right)$; the fluent of which appears, by Art. 5, to be $\frac{-2}{b(a+b)^{\frac{1}{2}}}\left(\alpha+\beta+\frac{3\gamma}{4}+\frac{3.5\delta}{4.6}+\frac{3.5.7\varepsilon}{4.6.8}, \&c.\right)$, which will vanish when $v=c$, and therefore needs no correction; and, when $v=1$, the series, without the factor, will be as follows:

$$2\alpha = 2\alpha$$

$$\beta = \frac{\sqrt{(1-cc)}}{2} + \frac{cc\alpha}{2}$$

$$\frac{3}{4}\gamma = \frac{3\sqrt{(1-cc)}}{4.4} + \frac{3.3cc\sqrt{(1-cc)}}{4.4.2} + \frac{3.3c^4\alpha}{4.4.2}$$

$$\frac{3.5}{4.6}\delta = \frac{3.5\sqrt{(1-cc)}}{4.6.6} + \frac{3.5.5cc\sqrt{(1-cc)}}{4.6.6.4} + \frac{3.5.5.3c^4\sqrt{(1-cc)}}{4.6.6.4.2}, \&c.$$

$$\frac{3.5.7}{4.6.8}\varepsilon = \frac{3.5.7\sqrt{(1-cc)}}{4.6.8.8} + \frac{3.5.7.7cc\sqrt{(1-cc)}}{4.6.8.8.6} + \frac{3.5.7.7.5c^4\sqrt{(1-cc)}}{4.6.8.8.6.4}, \&c.$$

&c. &c. &c. &c.

Now, the sum of the infinite series

$$\frac{1}{2}+\frac{3}{4.4}+\frac{3.5}{4.6.6}+\frac{3.5.7}{4.6.8.8}, \&c. \text{ being } = 2\,\text{H.L.}\,2 = \rho,$$

of $\frac{3.3}{4.4.2}+\frac{3.5.5}{4.6.6.4}+\frac{3.5.7.7}{4.6.8.8.6}+\frac{3.5.7.9.9}{4.6.8.10.10.8}$, &c. being $=\frac{3}{2}\text{H.L.}2=\sigma$,

of $\frac{3.5.5.3}{4.6.6.4.2}+\frac{3.5.7.7.5}{4.6.8.8.6.4}+\frac{3.5.7.9.9.7}{4.6.8.10.10.8.6}+\frac{3.5.7.9.11.11.9}{4.6.8.10.12.12.10.8}$, &c.

$=\frac{-1}{64}+\frac{41}{32}\text{H.L.}\,2=\tau$, &c. &c.

By proceeding as above, in Art. 9, a sufficiently near value of the whole series will be obtained in this expression, $\frac{32-10cc}{16-9cc}\alpha+\sqrt{(1-cc)}\,(\rho+\sigma cc+\tau c^4)$; and this, multiplied by its proper factor, gives

$$\frac{-2}{b(a+b)^{\frac{1}{2}}} \times \begin{cases} \frac{32-10cc}{16-9cc}\alpha \\ +\sqrt{(1-cc)}(\rho+\sigma cc+\tau c^4) \end{cases}$$ for the other part of the fluent sought. And since, by Art. 14, this fluent is $= B\pi$, we have, by dividing both sides by π,

$$B = \frac{2a}{b}A - \frac{2}{\pi b(a+b)^{\frac{1}{2}}} \times \begin{cases} \frac{32-10cc}{16-9cc}\alpha \\ +\sqrt{(1-cc)}(\rho+\sigma cc+\tau c^4), \end{cases}$$ which is its value when $n = \frac{3}{2}$.

17. We are next to find the value of this coefficient, when $n = \frac{5}{2}$; which, for the sake of distinction, I denote by B'.

With this value of n, the fluxionary expression in Art. 15, becomes $\frac{2a}{b(a+b)^{\frac{5}{2}}} \times \frac{2\dot{v}v^{-4}}{\sqrt{(1-vv)}\sqrt{(vv-cc)}} - \frac{2}{b(a+b)^{\frac{3}{2}}} \times \frac{2\dot{v}v^{-2}}{\sqrt{(1-vv)}\sqrt{(vv-cc)}}$; which being compared with the fluxions in Art. 5 and 10, it will appear that the fluent of the former part, when $v = 1$, is $= \frac{2a}{b}A'\pi$, and that the fluent of the latter part is $= \frac{-2}{b}A\pi$; which fluents, taken together, are, by Art. 14, $= B'\pi$. Therefore we have $B' = \frac{2a}{b}A' - \frac{2}{b}A = \frac{2}{b}(A'a - A)$.

Thirdly, *to find the Values of* C, D, E, *&c.*

18. The values of the coefficients A and B being now found, corresponding to the values of n $\frac{3}{2}$ and $\frac{5}{2}$, we might proceed in the same manner to find the value of C. For, if the equation in Art. 2, be multiplied by $2\cos. 2z$, and $\cos. (m-2)z + \cos. (m+2)z$ be written for $2\cos. 2z \times \cos. mz$, it will become $\frac{\dot{z}\times 2\cos. 2z}{(a-b\times\cos. z)^n} = \dot{z}(2A\times\cos. 2z + B(\cos. z + \cos. 3z) + C(1+\cos. 4z) + D(\cos. z+\cos. 5z)$, *&c.*) And the sum

of the fluents on the right-hand side, when $z = \pi$, will become barely $Cz = C\pi$. Therefore, the fluent of the left-hand side of the equation, when $z = \pi$, or of $\frac{-\dot{x}(4xx-2)}{\sqrt{(1-xx)}\,(a-bx)^n}$, when $x = 1$, or of $\frac{4aa-2bb-8a(a+b)vv+4(a+b)^2v^4}{bb(a+b)^n} \times \frac{2\dot{v}v^{1-2n}}{\sqrt{(1-vv)}\sqrt{(vv-cc)}}$, when $v = 1$, will be $= C\pi$. The fluent of this fluxion, it is evident, will consist of three parts, the first and second of which, n being $= \frac{3}{2}$, are obviously attainable from the values of A and B above found in Art. 9. and 16.; and the third in series similar to those which have been given in the former part of this paper.

It is evident also that, if n be $= \frac{5}{2}$, all three parts of this fluent are attainable from the values of the two coefficients already found, and C' would be $= -2A' + \frac{2}{b}(B'a - B)$.

19. And in this manner may the other coefficients, D, E, F, &c. be determined. And since the cosines of $3z$, $4z$, &c. are $= 4x^3 - 3x$, $8x^4 - 8x^2 + 1$, &c. respectively; and since $x = \frac{a-(a+b)vv}{b}$, it is evident that the numerator of the fraction into which the fluxion in the preceding article is to be multiplied, will be always of this form, *viz.* $p + qvv + rv^4 + sv^6$, &c.; from which it follows, that, if the values of A', A, $\underset{\prime}{A}$, &c. corresponding to n, $n-1$, $n-2$, &c. be computed, the values of C, D, E, F, and all the rest, may be found in terms of A', A, $\underset{\prime}{A}$, &c. with the coefficients a and b. But, since the easiest method, that has come to my hands, of computing the values of C, D, E, &c. after A and B are found, is explained in M. EULER's *Institutiones Calculi integralis*, Vol. I. p. 181,* I shall

* The coefficients 2, 3, and 4, after 2 C *sin.* 3 D *sin.* and 4 E *sin.* in line 8 of the page above referred to, are wanting; and $-$ is printed for $+$ before 2 C, in line 13. And there are press errors in many other places. It is to be regretted, that so excellent a book was not more correctly printed.

not pursue this method any further; but, having examined his process, and corrected the errors of the press which occur in it, now give the equations expressing the values of C, D, E, F, *&c.* which were obtained by that method.

20. For the sake of brevity, let $\frac{a}{b} = d$; then will the general values of C, D, E, F, *&c.* be expressed by these equations:

$$C = \frac{2nA - 2dB}{n-2}$$

$$D = \frac{(n+1)B - 4dC}{n-3}$$

$$E = \frac{(n+2)C - 6dD}{n-4}$$

$$F = \frac{(n+3)D - 8dE}{n-5}$$

&c.

where the law of continuation is very obvious. And the particular values of these letters, when $n = \frac{1}{2}, \frac{3}{2}$, and $\frac{5}{2}$, will be as expressed in the following columns:

	$n = \frac{1}{2}$	$n = \frac{3}{2}$	$n = \frac{5}{2}$
$C =$	$\frac{4d}{3}B - \frac{4}{3}A$	$\frac{4d}{1}B - 6A$	$-4dB + 10A$
$D =$	$\frac{8d}{5}C - \frac{3}{5}B$	$\frac{8d}{3}C - \frac{5}{3}B$	$\frac{8d}{1}C - \frac{7}{1}B$
$E =$	$\frac{12d}{7}D - \frac{5}{7}C$	$\frac{12d}{5}D - \frac{7}{5}C$	$\frac{12d}{3}D - \frac{9}{3}C$
$F =$	$\frac{16d}{9}E - \frac{7}{9}D$	$\frac{16d}{7}E - \frac{9}{7}D$	$\frac{16d}{5}E - \frac{11}{5}D$
	&c.	*&c.*	*&c.*

21. The solution of the problem being now finished, it may perhaps be satisfactory to the reader to see how the sums of the very slowly converging numerical series, which arose in Art. 7, 11, and 16, were obtained; the investigations of which,

because they would have detained him too long from the immediate subject of this paper, if they had been inserted in it, are given in the following Appendix.

AN APPENDIX TO THE FOREGOING PAPER:

In which the Method of obtaining the Sums of the very slowly converging numerical Series which are used therein, and of many others of that Kind which arise in the Fluents of Binomial Surds, is explained and illustrated; and some Observations, tending to facilitate and abridge the Computations of the Coefficients A *and* B, *are added.*

1. As the sums of the very slowly converging numerical series, which arose in Art. 7, 11, and 16, of the preceding paper, are not exhibited in any book that has come to my hands, and as series of that kind frequently occur, I conceive that the following method of obtaining their sums will be acceptable to the lovers of mathematics in general, and particularly to those who have frequent occasion to use the sums of such series. And, having observed, while considering the literal expressions in the preceding paper for the values of A and B, that others, no less accurate, might be derived from them, by which the arithmetical operations would be facilitated and abridged, I thought these observations might likewise be acceptable to those who are engaged in the theory of astronomy, and have inserted them also in this paper; which, therefore, consists of two principal parts, the summation of the slowly converging series, and the observations now mentioned.

I. *The Summation of the slowly converging Series.*

2. But, before I begin the investigation, it will be proper to premise a few particulars, an attention to which will shorten and facilitate the operations now to be performed.

1st. That $\frac{1-\sqrt{(1-yy)}}{1+\sqrt{(1-yy)}}$ being $=\frac{1-\sqrt{(1-yy)}}{1+\sqrt{(1-yy)}}\times\frac{1+\sqrt{(1-yy)}}{1+\sqrt{(1-yy)}}$, is $=\left(\frac{y}{1+\sqrt{(1-yy)}}\right)^2$; from which it follows, that H. L. of $\frac{1-\sqrt{(1-yy)}}{1+\sqrt{(1-yy)}}$ is $=2$ H. L. $\frac{y}{1+\sqrt{(1-yy)}}$.

2dly. That the fluxion of H. L. $\frac{2}{1+\sqrt{(1-yy)}}$ is $=\frac{\dot{y}}{y\sqrt{(1-yy)}}-\frac{\dot{y}}{y}$. For it is $=$ the fluxion of $-$ H. L. $\left(1+\sqrt{(1-yy)}\right)$ $=\frac{y\dot{y}}{\sqrt{(-yy)}}\times\frac{1}{1+\sqrt{(1-yy)}}$; and, if both numerator and denominator of this expression be multiplied by $1-\sqrt{(1-yy)}$, it will become $\frac{y\dot{y}}{\sqrt{(1-yy)}}\times\frac{1-\sqrt{(1-yy)}}{yy}$, which is $=\frac{\dot{y}}{y\sqrt{(1-yy)}}$ $-\frac{\dot{y}}{y}$.

3dly. That the H. L. $\frac{2}{1+\sqrt{(1-yy)}}$ is therefore $=\int\frac{\dot{y}}{y\sqrt{(1-yy)}}$ $-\int\frac{\dot{y}}{y}=\frac{yy}{2.2}+\frac{3y^4}{2.4.4}+\frac{3.5y^6}{2.4.6.6}+\frac{3.5.7y^8}{2.4.6.8.8}$, &c.

4thly. That, Q being put $=\sqrt{(1-yy)}$, the fluxion of $\frac{Q}{y^n}$ will be $=\frac{\dot{y}}{Q}\left(\frac{-n}{y^{n+1}}+\frac{n-1}{y^{n-1}}\right)$. For it will be $\frac{\dot{Q}}{y^n}-\frac{n\dot{y}Q}{y^{n+1}}$ $=\frac{-y\dot{y}}{y^nQ}-\frac{n\dot{y}Q^2}{y^{n+1}Q}=\frac{-\dot{y}}{y^{n-1}Q}-\frac{n\dot{y}(1-yy)}{y^{n+1}Q}=\frac{-n\dot{y}}{y^{n+1}Q}+\frac{(n-1)\dot{y}}{y^{n-1}Q}$ $=\frac{\dot{y}}{Q}\left(\frac{-n}{y^{n+1}}+\frac{n-1}{y^{n-1}}\right)$.

5thly. That, when any quantities, as $\boxed{\dot{u}\left(\frac{f}{y^4}+\frac{g}{yy}+h\right)}$, are circumscribed by a parallelogram, it denotes that a substi-

tution for these quantities has been made in the same equation in which it occurs, and consequently that they are no longer to be considered as part of that equation. This I have found to be better than cancelling, as it answers the same end without obliteration.

We may now proceed to the summation of the series before-mentioned, in which the utility of what has been premised will quickly appear.

3. As it does not seem necessary to set down the operations of computing the sums of all the series which arose in the preceding paper, I will make choice of the summation of those which occur in Art. 11, they being the most difficult, as the properest examples to illustrate this method.

It is well known that the expression $\frac{2\dot{y}y^{-5}}{\sqrt{(1-yy)}}$ is $= 2\dot{y}y^{-5} + \frac{2\dot{y}y^{-3}}{2} + \frac{2.3\dot{y}y^{-1}}{2.4} + \frac{2.3.5\dot{y}y}{2.4.6} + \frac{2.3.5.7\dot{y}y^3}{2.4.6.8} + \frac{2.3.5.7.9\dot{y}y^5}{2.4.6.8.10}$, &c. from which equation we have $\frac{2\dot{y}y^{-5}}{(1-yy)} - 2\dot{y}y^{-5} - \dot{y}y^{-3} - \frac{3\dot{y}y^{-1}}{4} = \frac{3.5\dot{y}y}{4.6} + \frac{3.5.7\dot{y}y^3}{4.6.8} + \frac{3.5.7.9\dot{y}y^5}{4.6.8.10}$, &c. Now the fluents of the terms on the first side are

$$\left\{\begin{array}{l} \frac{-\sqrt{(1-yy)}}{2y^4} - \frac{3\sqrt{(1-yy}}{4y^2} \boxed{+\frac{3}{4}\text{H.L.}\frac{y}{1+\sqrt{(1-yy)}}} \\ +\frac{1}{2y^4} + \frac{1}{2yy} \boxed{-\frac{3}{4}\text{H.L.}y} + \frac{3}{4}\text{H.L.}\frac{1}{1+\sqrt{(1-yy)}}; \end{array}\right.$$

on the second side, the fluents are $\frac{3.5yy}{4.6.2} + \frac{3.5.7y^4}{4.6.8.4} + \frac{3.5.7.9y^6}{4.6.8.10.6}$, &c. And, to find whether these two expressions are = each other, or have a constant difference, we may compute their numerical values, y being put = any small simple fraction, such as $\frac{1}{10}$, $\frac{1}{100}$, or $\frac{1}{1000}$, either of which values of y is a very convenient one for the purpose. But an easier method to dis-

cover the constant quantities which lie concealed in some of the terms on the first side, is to convert that side into series, by the binomial theorem; which will then be as follows:

$$\begin{aligned}
\frac{-\surd(1-yy)}{2y^4} &= -\tfrac{1}{2}y^{-4}+\tfrac{1}{4}y^{-2}+\tfrac{1}{16}+\tfrac{1}{32}y^2+\tfrac{5}{256}y^4, \text{ \&c.}\\
\frac{-3\surd(1-yy)}{4yy} &= \qquad\quad -\tfrac{3}{4}y^{-2}+\tfrac{3}{8}+\tfrac{3}{32}y^2+\tfrac{3}{64}y^4, \text{ \&c.}\\
+\frac{1}{2y^4}+\frac{1}{2yy} &= +\tfrac{1}{2}y^{-4}+\tfrac{1}{2}y^{-2}\\
+\tfrac{3}{4}\text{H.L.}\frac{1}{1+\surd(1-yy)} &= \qquad\qquad -\tfrac{3}{4}\text{H.L.}2+\tfrac{3}{16}y^2+\tfrac{9}{128}y^4, \text{\&c.}
\end{aligned}$$

The sum is $= \quad * \quad * \quad +\tfrac{7}{16}-\tfrac{3}{4}\text{H.L.}2, +\tfrac{5}{16}y^2+\tfrac{35}{256}y^4$, &c. which evidently differs from the series on the second side by the constant quantity $\tfrac{7}{16}-\tfrac{3}{4}$ H.L. 2. We therefore have, by subtracting this constant quantity from the first side,

$$\left.\begin{array}{l}\dfrac{-\surd(1-yy)}{2y^4}-\dfrac{3\surd(1-yy)}{4yy}+\tfrac{3}{4}\text{H.L.}\dfrac{2}{1+\surd(1-yy)}\\[2ex] +\dfrac{1}{2y^4}+\dfrac{1}{2yy}-\dfrac{7}{16}\end{array}\right\}=\frac{3\cdot5yy}{4.6.2}+\frac{3\cdot5\cdot7y^4}{4.6.8.4}+\frac{3\cdot5\cdot7\cdot9y^6}{4.6.8.10.4}, \text{\&c.}$$

which, when y becomes $= 1$, becomes

$$\left.\begin{array}{l} * \qquad * \;+\tfrac{3}{4}\text{H.L.}2\\ \boxed{+\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{7}{16}}+\tfrac{9}{16}\end{array}\right\}=\frac{3\cdot5}{4.6.2}+\frac{3\cdot5\cdot7}{4.6.8.4}+\frac{3\cdot5\cdot7\cdot9}{4.6.8.10.6}, \text{\&c.}$$

which is the series denoted by λ' in Art. 12. of the preceding paper.

4. If the equation of fluents in the preceding Article be divided by y, and if $\frac{3\cdot5y}{4.6.2}=\frac{5}{16}y$ be then taken from both sides of it, and u be written for H.L. $\frac{2}{1+\surd(1-yy)}$, we shall have

$$\left.\begin{array}{l}\dfrac{-\surd(1-yy)}{2y^5}-\dfrac{3\surd(1-yy)}{4y^3}+\dfrac{3u}{4y}\\[2ex] +\dfrac{1}{2y^5}+\dfrac{1}{2y^3}-\dfrac{7}{16y}-\dfrac{5y}{16}\end{array}\right\}=\frac{3\cdot5\cdot7y^3}{4.6.8.4}+\frac{3\cdot5\cdot7\cdot9y^5}{4.6.8.10.6}+\frac{3\cdot5\cdot7\cdot9\cdot11y^7}{4.6.8.10.12.8}, \text{\&c.}$$

And, if this equation be put into fluxions, and Q be written for $\surd(1-yy)$, for the sake of brevity, there will be

$$\left.\begin{array}{l}\frac{5}{2y^6}+\frac{9}{4y^4}+\frac{3}{4yy}\\ \quad -\frac{4}{2y^4}-\frac{6}{4yy}\end{array}\right\}\frac{\dot{y}}{Q}\boxed{+\frac{3\dot{u}}{4y}}-\frac{3u\dot{y}}{4yy}$$

$$+\left(-\frac{5}{2y^6}+\frac{3}{2y^4}\begin{array}{l}+\frac{7}{16yy}\\ -\frac{3}{4yy}\end{array}-\frac{5}{16}\right)\dot{y}=\frac{3.5.7.3\dot{y}yy}{4.6.8.4}+\frac{3.5.7.9.5\dot{y}y^4}{4.6.8.10.6}$$

$$+\frac{3.5.7.9.11.7\dot{y}y^6}{4.6.8.10.12.8}, \text{\&c.}$$

And this equation, more concisely expressed and divided by y, gives

$$\left(\frac{5}{2y^7}+\frac{1}{4y^5}-\frac{3}{4y^3}\right)\frac{\dot{y}}{Q}-\frac{3u\dot{y}}{4y^3}$$

$$+\left(\frac{-5}{2y^7}-\frac{3}{2y^5}-\frac{5}{16y^3}-\frac{5}{16y}\right)\dot{y}=\frac{3.5.7.3\dot{y}y}{4.6.8.4}+\frac{3.5.7.9.5\dot{y}y^3}{4.6.8.10.6}+\frac{3.5.7.9.11.7\dot{y}y^5}{4.6.8.10.12.8}, \text{\&c.}$$

Now the fluent of the series on the second side of this equation is found, by the methods which have been long known, to be $\frac{3.5.7.3yy}{4.6.8.4.2}+\frac{3.5.7.9.5y^4}{4.6.8.10.6.4}+\frac{3.5.7.9.11.7y^6}{4.6.8.10.12.8.6}$, *&c.* and the fluent of the terms on the first side will be very easily obtained, by the following assumption, and attention to what was shewn in Art. 2. of this paper.

For the fluent of the terms on the first side of this equation, assume

$$\left(\frac{a}{y^6}+\frac{b}{y^4}+\frac{c}{y^2}\right)Q+u\left(\frac{f}{yy}+g\right)$$

$+\frac{p}{y^6}+\frac{q}{y^4}+\frac{r}{yy}+s\,\text{H. L.}\,y$; then will the fluxion of this expression be

$$\left.\begin{array}{l}\frac{-6a}{y^7}-\frac{4b}{y^5}-\frac{2c}{y^3}\\ \quad +\frac{5a}{y^5}+\frac{3b}{y^3}+\frac{c}{y}\\ \quad\quad +\frac{f}{y^3}+\frac{g}{y}\end{array}\right\}\frac{\dot{y}}{Q}\begin{array}{l}\boxed{+\dot{u}\left(\frac{f}{yy}+g\right)}\\ -\frac{2fu\dot{y}}{y^3}\end{array}$$

$$+\left.\begin{matrix}\left(\dfrac{-6p}{y^7}-\dfrac{4q}{y^5}-\dfrac{2r}{y^3}+\dfrac{s}{y}\right.\\ \qquad\qquad -\dfrac{f}{y^3}-\dfrac{g}{y}\end{matrix}\right\}\dot{y},$$ which being put = the first side of the foregoing equation, there will arise as many simple equations for determining the coefficients a, b, c, *&c.* as there are letters of that kind in the assumed fluent, from which their values will easily be found. For there will be

$$\begin{array}{ll}
6a = \frac{-5}{2}, \text{ from which} & a = \frac{-5}{12}, \\
4b = 5a - \frac{1}{4}, & b = \frac{-7}{12}, \\
2c = 3b + f + \frac{3}{4}, & c = \frac{-5}{16}, \\
2f = \frac{3}{4}, & f = \frac{3}{8}, \\
g = -c, & g = \frac{5}{16}, \\
6p = \frac{5}{2}, & p = \frac{5}{12}, \\
4q = \frac{3}{2}, & q = \frac{3}{8}, \\
2r = \frac{5}{16} - f, & r = \frac{-1}{32}, \\
s = g - \frac{5}{16}, & s = 0.
\end{array}$$

The variable part, therefore, of the fluent of the first side of the above equation is

$$Q\left(\frac{-5}{12y^6}-\frac{7}{12y^4}-\frac{5}{16yy}\right)+u\left(\frac{3}{8yy}+\frac{5}{16}\right)$$
$$+\frac{5}{12y^6}+\frac{3}{8y^4}-\frac{1}{32yy}.$$

Now, to discover the constant quantities which lie concealed in this expression, we must proceed as above in Art. 3.

$$\frac{-5\sqrt{(1-yy)}}{12y^6} \text{ is} = -\frac{5}{12}y^{-6} + \frac{5}{12.2}y^{-4} + \frac{5}{12.8}y^{-2} + \frac{5}{12.16} + \frac{5.5}{12.8.16}y^2, \&c.$$

$$\frac{-7\sqrt{(1-yy)}}{12y^4} = -\frac{7}{12}y^{-4} + \frac{7}{12.2}y^{-2} + \frac{7}{12.8} + \frac{7}{12.16}y^2, \&c.$$

$$\frac{-5\sqrt{(1-yy)}}{16yy} = -\frac{5}{16}y^{-2} + \frac{5}{16.2} + \frac{5}{16.8}y^2, \&c.$$

$$u\left(\frac{3}{8yy} + \frac{5}{16}\right)^* = +\frac{3}{4.8} + \frac{29}{8.32}y^2, \&c.$$

To which add the other terms - $\left.\right\}$ $+\frac{5}{12}y^{-6} + \frac{3}{8}y^{-4} - \frac{1}{32}y^{-2}$

The sum is $\quad * \quad * \quad * + \frac{67}{192} + \frac{105}{512}y^2, \&c.$

which exceeds the series above found, by the constant quantity $\frac{67}{192}$. We therefore now have

$$Q\left(\frac{-5}{12y^6} - \frac{7}{12y^4} - \frac{5}{16yy}\right) + u\left(\frac{3}{8yy} + \frac{5}{16}\right) + \frac{5}{12y^6} + \frac{3}{8y^4} - \frac{1}{32yy} - \frac{67}{192} = \frac{3.5.7.3\,yy}{4.6.8.4.2} + \frac{3.5.7.9.5\,y^4}{4.6.8.10.6.4} + \frac{3.5.7.9.11.7\,y^6}{4.6.8.10.12.8.6}, \&c.;$$

and when y becomes $= 1$, Q being then $= 0$, and $u =$ H. L. 2, this equation becomes

$$\left.\begin{matrix} \text{H.L. } 2\left(\frac{3}{8} + \frac{5}{16}\right) \\ + \frac{5}{12} + \frac{3}{8} - \frac{1}{32} - \frac{67}{192} \end{matrix}\right\} = \left.\begin{matrix} \frac{11}{16}\text{H.L. } 2 \\ + \frac{79}{192} \end{matrix}\right\} = \frac{3.5.7.3}{4.6.8.4.2} + \frac{3.5.7.9.5}{4.6.8.10.6.4} + \frac{3.5.7.9.11.7}{4.6.8.10.12.8.6}, \&c.$$

which is the value of μ' in Art. 12. of the preceding paper.

5. If the last literal equation be divided by y, and $\frac{3.5.7.3y}{4.6.8.4.2} = \frac{105y}{512}$ be then taken from both sides, we shall have

$$\left.\begin{matrix} Q\left(\frac{-5}{12y^7} - \frac{7}{12y^5} - \frac{5}{16y^3}\right) + u\left(\frac{3}{8y^3} + \frac{5}{16y}\right) \\ + \frac{5}{12y^7} + \frac{3}{8y^5} - \frac{1}{32y^3} - \frac{67}{192y} - \frac{105y}{512} \end{matrix}\right\} =$$

* $u\left(\frac{3}{8yy} + \frac{5}{16}\right)$ is $= \left(\frac{1}{4}yy + \frac{3}{32}y^4, \&c.\right)\left(\frac{3}{8}y^{-2} + \frac{5}{16}\right)$. See Art. 2.

$$\frac{3.5.7.9.5y^3}{4.6.8.10.6.4}+\frac{3.5.7.9.11.7y^5}{4.6.8.10.12.8.6.}+\frac{3.5.7.9.11.13.9y^7}{4.6.8.10.12.14.10.8}, \text{ \&c.}$$

which equation, in fluxions, gives

$$\left.\begin{array}{l}\frac{5.7}{12y^8}+\frac{7.5}{12y^6}+\frac{5.3}{16y^4}\\ \quad-\frac{5.6}{12y^6}-\frac{7.4}{12y^4}-\frac{5.2}{16yy}\\ \quad\quad+\frac{3}{8y^4}+\frac{5}{16yy}\end{array}\right\}\frac{\dot{y}}{Q}\quad \begin{array}{l}\boxed{+\dot{u}\left(\frac{3}{8y^3}+\frac{5}{16y}\right)}\\ -u\dot{y}\left(\frac{9}{8y^4}+\frac{5}{16yy}\right)\end{array}$$

$$+\left(\begin{array}{l}-\frac{5.7}{12y^8}-\frac{3.5}{8y^6}+\frac{3}{32y^4}+\frac{67}{192yy}-\frac{105}{512}\\ \quad\quad-\frac{3}{8y^4}-\frac{5}{16yy}\end{array}\right\}\dot{y}=$$

$$\frac{3.5.7.9.5.3\dot{y}yy}{4.6.8.10.6.4}+\frac{3.5.7.9.11.7.5\dot{y}y^4}{4.6.8.10.12.8.6}+\frac{3.5.7.9.11.13.9.7\dot{y}y^6}{4.6.8.10.12.14.10.8}, \text{ \&c.}$$

And this equation, more concisely expressed, and divided by y, gives

$$\frac{\dot{y}}{Q}\left(\frac{35}{12y^9}+\frac{5}{12y^7}-\frac{49}{48y^5}-\frac{5}{16y^3}\right)-u\dot{y}\left(\frac{9}{8y^5}+\frac{5}{16y^3}\right)$$

$$+\dot{y}\left(-\frac{35}{12y^9}-\frac{15}{8y^7}-\frac{9}{32y^5}+\frac{7}{192y^3}-\frac{105}{512y}\right)=$$

$$\frac{3.5.7.9.5.3\dot{y}y}{4.6.8.10.6.4}+\frac{3.5.7.9.11.7.5\dot{y}y^3}{4.6.8.10.12.8.6}+\frac{3.5.7.9.11.13.9.7\dot{y}y^5}{4.6.8.10.12.14.10.8}, \text{ \&c.}$$

Now the fluent of the fluxionary series on the second side of the equation being obviously the series $\frac{3.5.7.9.5.3yy}{4.6.8.10.6.4.2}+\frac{3.5.7.9.11.7.5y^4}{4.6.8.10.12.8.6.4}+\frac{3.5.7.9.11.13.9.7y^6}{4.6.8.10.12.14.10.8.6}$, &c. we are next to take the fluent of the expression on the first side, and to correct it, that it may be = this series; which may be done as follows:

For the fluent sought, assume

$$Q\left(\frac{a}{y^8}+\frac{b}{y^6}+\frac{c}{y^4}+\frac{d}{yy}\right)+u\left(\frac{f}{y^4}+\frac{g}{yy}+h\right)$$

$$+\frac{p}{y^8}+\frac{q}{y^6}+\frac{r}{y^4}+\frac{s}{yy}+t\text{ H. L. }y,$$ and take the fluxion of this expression, which will be

$$\left.\begin{array}{r}\frac{-8a}{y^9}-\frac{6b}{y^7}-\frac{4c}{y^5}-\frac{2d}{y^3}\\ +\frac{7a}{y^7}+\frac{5b}{y^5}+\frac{3c}{y^3}+\frac{d}{y}\\ +\frac{f}{y^5}+\frac{g}{y^3}+\frac{h}{y}\end{array}\right\}\frac{\dot{y}}{Q}\;\begin{array}{l}\boxed{+\dot{u}\left(\frac{f}{y^4}+\frac{g}{yy}+h\right)}\\ -u\dot{y}\left(\frac{4f}{y^5}+\frac{2g}{y^3}\right)\end{array}$$

$$+\left.\left(\begin{array}{r}\frac{-8p}{y^9}-\frac{6q}{y^7}-\frac{4r}{y^5}-\frac{2s}{y^3}+\frac{t}{y}\\ -\frac{f}{y^5}-\frac{g}{y^3}-\frac{h}{y}\end{array}\right.\right\}\dot{y};$$ and these fluxionary terms being put $=$ to those on the first side of the preceding equation, there will arise

$8a=\frac{-35}{12}$,	from which	$a=\frac{-35}{8.12}$,
$6b=7a-\frac{5}{12}$,		$b=\frac{-95}{2.8.12}$,
$4c=5b+f+\frac{49}{48}$,		$c=\frac{-75}{4.8.8}$,
$2d=3c+g+\frac{5}{16}$,		$d=\frac{-105}{8.8.8}$,
$4f=\frac{9}{8}$,		$f=\frac{9}{32}$,
$2g=\frac{5}{16}$,		$g=\frac{5}{32}$,
$h=-d$,		$h=\frac{105}{8.8.8}$,
$8p=\frac{35}{12}$,		$p=\frac{35}{8.12}$,
$6q=\frac{15}{8}$,		$q=\frac{5}{16}$,
$4r=\frac{9}{32}-f$,		$r=0$,
$2s=\frac{-7}{192}-g$,		$s=\frac{-37}{4.8.12}$,
$t=\frac{-105}{512}+h$,		$t=0$.

Our assumed fluent, therefore, is

$$Q\left(\frac{-35}{8.12y^8}-\frac{95}{12.16y^6}-\frac{75}{4.8.8y^4}-\frac{105}{8.8.8yy}\right)+u\left(\frac{9}{32y^4}+\frac{5}{32yy}+\frac{105}{8.8.8}\right)$$

$+\frac{35}{8.12y^8}+\frac{5}{16y^6}\quad * \quad -\frac{37}{4.8.12yy}\quad *$, which may be corrected in the manner shewn in the two preceding Articles, or more expeditiously, as follows.

It is pretty evident, from the correction of the fluent in the preceding Article, that the constant quantities which lie concealed in this fluent, will appear in those terms only, (when the radical quantity $\sqrt{(1-yy)}$, and the logarithm u, is expressed in series,) in which the index of y is o. Thus, the constant quantities will appear as below.

The 5th term of $\frac{-35\sqrt{(1-yy)}}{8.12y^8}$ is $\frac{35}{8.12}\times\frac{5y^8}{8.16y^8}=\frac{175}{4.12.16.16}$;

The 4th term of $\frac{-95\sqrt{(1-yy)}}{12.16y^6}$ is $\frac{95}{12.16}\times\frac{y^6}{16y^6}=\frac{95}{12.16.16}$;

The 3d term of $\frac{-75\sqrt{(1-yy)}}{4.8.8y^4}$ is $\frac{75}{4.8.8}\times\frac{y^4}{8y^4}=\frac{75}{8.16.16}$;

The 2d term of $\frac{-105\sqrt{(1-yy)}}{8.8.8yy}$ is $\frac{105}{8.8.8}\times\frac{yy}{2yy}=\frac{105}{4.16.16}$;

and the terms in which the index of y is o, in the logarithmic part, *viz.* $\left(\frac{1}{4}yy+\frac{3}{2.16}y^4, \text{\&c.}\right)\left(\frac{9}{2.16}y^{-4}+\frac{5}{2.16}y^{-2}+\frac{105}{8.8.8}\right)$, are these two, - - $\frac{27y^4}{4.16.16y^4}=\frac{27}{4.16.16}$,

and $\frac{5yy}{8.16yy}=\frac{5}{8.16}$.

The sum of these six fractions is $\frac{1023}{4096}$. The equation of fluents therefore is

$$Q\left(\frac{-35}{8.12y^8}-\frac{95}{12.16y^6}-\frac{75}{16.16y^4}-\frac{105}{8.8.8yy}\right)+u\left(\frac{9}{32y^4}+\frac{5}{32yy}+\frac{105}{8.8.8}\right)$$

$+\frac{35}{8.12y^8}+\frac{5}{16y^6}\quad * \quad -\frac{37}{4.8.12yy}-\frac{1023}{4096}=$ the series

$$\frac{3.5.7.9.5.3\,yy}{4.6.8.10.6.4.2} + \frac{3.5.7.9.11.7.5\,y^4}{4.6.8.10.12.8.6.4} + \frac{3.5.7.9.11.13.9.7\,y^6}{4.6.8.10.12.14.10.8.6}, \text{ \&c.}$$

which, when $y = 1$, becomes

$$\left.\begin{array}{l} \text{H. L. } 2\left(\frac{9}{32} + \frac{5}{32} + \frac{105}{8.8.8}\right) \\ + \frac{35}{8.12} + \frac{5}{16} - \frac{37}{4.8.12} - \frac{1023}{4096} \end{array}\right\} = \frac{4067}{12288} + \frac{329}{512}\text{ H. L. } 2$$

$= \frac{3.5.7.9.5.3}{4.6.8.10.6.4.2} + \frac{3.5.7.9.11.7.5}{4.6.8.10.12.8.6.4} + \frac{3.5.7.9.11.13.9.7}{4.6.8.10.12.14.10.8.6}$, *&c.* which is the value of v' in Art. 12. of the foregoing paper.

These three examples, I conceive, are sufficient to illustrate this method of summing the slowly converging numerical series which arose in the solution of the problem in the preceding paper. The three series of which the sums are now investigated are, as was before observed, the most difficult to sum of all that arose in that solution; so that, whoever understands what is done here, may, with great ease, compute the sums of the rest of the series which are found there, and of many others of this kind, which arise in the solution of problems.

II. *Observations, tending to facilitate and abridge the numerical Computations of* A *and* B *in the preceding Paper.*

6. The radical factor $\sqrt{(1-cc)}$, in the literal expressions of the values of A and B, may be taken away, by multiplying the other factors by its equivalent $1 - \frac{cc}{2} - \frac{c^4}{8} - \frac{c^6}{16}$, *&c.* in consequence of which, other expressions will be obtained, better adapted to the purpose of numerical calculation. This will appear by the following operations.

The product of $\sqrt{(1-cc)} \times$ the other factor in the expression of the value of A, in Art. 9. of the preceding paper, will be

$$\frac{\dfrac{2}{cc}+\lambda+\mu cc+\nu c^4}{1-\dfrac{cc}{2}-\dfrac{c^4}{8}-\dfrac{c^6}{16},\ \&c.}$$

$$\left.\begin{array}{l}\frac{2}{cc}+\lambda+\quad\mu cc+\quad\nu c^4\\ \quad-1-\frac{1}{2}\lambda cc-\frac{1}{2}\mu c^4\\ \quad\quad-\frac{1}{4}\ cc-\frac{1}{8}\lambda c^4\\ \quad\quad\quad\quad-\frac{1}{8}\ c^4\end{array}\right\}=\frac{2}{cc}+e+fcc+gc^4,\ \&c.$$

where e, f, and g, are $=\lambda-1$, $\mu-\frac{1}{2}\lambda-\frac{1}{4}$, and $\nu-\frac{1}{2}\mu-\frac{1}{8}\lambda-\frac{1}{8}$, respectively; in numbers $=$ 0·1931472, 0·1036802, and 0·0687064, respectively. And this expression, which is evidently more simple than the former, is somewhat nearer than that to the value of the whole series, as will appear to any one who shall compute the value of the next coefficient.

7. In like manner, the product of the two factors in the value of A′, in Art. 13. will be

$$\frac{\dfrac{4}{3c^4}+\dfrac{5}{3cc}+\lambda'+\mu'cc+\nu'c^4}{1-\dfrac{cc}{2}-\dfrac{c^4}{8}-\dfrac{c^6}{16}-\dfrac{5c^8}{8.16},\ \&c.}$$

$$\left.\begin{array}{l}\frac{4}{3c^4}+\frac{5}{3cc}+\lambda'+\mu'cc+\quad\nu'c^4\\ \quad-\frac{2}{3cc}-\frac{5}{6}-\frac{1}{2}\lambda'cc-\frac{1}{2}\mu'c^4\\ \quad\quad\quad-\frac{1}{6}-\frac{5}{24}\ cc-\frac{1}{8}\lambda'c^4\\ \quad\quad\quad\quad-\frac{1}{12}\ cc-\frac{5}{48}\ c^4\\ \quad\quad\quad\quad\quad\quad-\frac{5}{96}\ c^4\end{array}\right\}=\frac{4}{3c^4}+\frac{1}{cc}+h+icc+kc^4,\ \&c.$$

which expression also is more simple than that from which it is derived, while the accuracy of it is not less, as is pretty evident on inspection. And, that the numerical values of h, i, and k, are very easily attainable from the values of λ', μ', and ν',

given above in Art. 3, 4, and 5, of this paper, is very obvious. In a subsequent Article, these values will be inserted.

8. And the product of the two factors in the value of B, in Art. 16, may also be exchanged for a more convenient expression, by a like process.

$$\begin{array}{l} \rho + \sigma cc + \tau c^4 \\ 1 - \frac{cc}{2} - \frac{c^4}{8}, \text{\&c.} \\ \hline \left.\begin{array}{l} \rho + \sigma cc + \tau c^4 \\ \quad - \frac{1}{2}\rho cc - \frac{1}{2}\sigma c^4 \\ \qquad\qquad - \frac{1}{8}\rho c^4 \end{array}\right\} = \rho + lcc + mc^4, \text{\&c.} \end{array}$$

which expression also is more accurate than that from which it is derived, as well as more simple. The numerical values of l and m, which are evidently given from those of ρ, σ, and τ, will be inserted a little further on, when we come to an example of calculating the values of A and B in numbers.

9. The numerical calculation of the other member also, in which α enters, may be facilitated and abridged, by the following considerations.

If c be put for the sine of an angle, radius being 1, then will $1+\sqrt{(1-cc)}$ be the versed-sine of the supplement of that angle, and $\frac{c}{1+\sqrt{(1-cc)}}$ will be = the tangent of half that angle; from which it follows, that the reciprocal of this quantity, *viz.* $\frac{1+\sqrt{(1-cc)}}{c}$, is = the co-tangent of half the angle of which the sine is c. The common logarithm of $\frac{1+\sqrt{(1-cc)}}{c}$ may therefore be taken out from TAYLOR's excellent Tables,* and quickly converted into an hyperbolic logarithm, by Table XXXVII. of DODSON's Calculator.

* These valuable Tables are computed to every second of the quadrant.

10. An expression of this kind, $\frac{p \pm rcc}{q \pm scc}$, when c is the only variable quantity, consisting of several figures, and r and s are likewise long numbers, will be much better adapted to the use of logarithms, when put in this form, $\frac{r}{s} \times \frac{p \div r \pm cc}{q \div s \pm cc}$; because the multiplications of r and s into cc, or additions of their logarithms and taking out two numbers, are by this means exchanged for the addition of the constant logarithm of $\frac{r}{s}$: the quotients $\frac{p}{r}$ and $\frac{q}{s}$, once found, being constant numbers. Thus, the numerical value of even $\frac{8-2cc}{8-5cc}$, where r and s are single figures, is more easily obtained by $\frac{4}{10} \times \frac{4-cc}{1 \cdot 6-cc}$, than by the former expression.

11. But it will appear upon trial, that the arithmetical value of any three terms $p' + q'cc + r'c^4$, in which p', q', and r', are constant quantities, and cc consists of five or six places of figures, may, in general, be more easily obtained by logarithms than the arithmetical value of $\frac{r}{s} \times \frac{p \div r \pm cc}{q \div s \pm cc}$. And, since the difference of the values of these two expressions is inconsiderable in the present case, I shall make no further use of the fractional expression; but observe, that the logarithm of $q'cc$, in the other expression, being found, the logarithm of $r'c^4$ will be had, by adding to it the logarithm of $\frac{r'}{q'}cc$; for $q'cc \times \frac{r'}{q'}cc = r'c^4$. And, since the logarithms of the numbers which stand in the places of q' and $\frac{r'}{q'}$ may be taken out and reserved for use, and the logarithms of cc and α, once found, will serve for all the terms in which these quantities occur, it will appear by an example, that neither many logarithms, nor many numbers correspond-

ing to logarithms, need be taken out of other tables, in computing the value of A or B.

12. It will now be proper, since the literal expressions of the values of A and B have been exchanged for others which are more convenient, to bring the new equations together in one view, and, after that, to give an example of the numerical calculations by them.

It appears, by Art. 9, 10, 13, 16, and 17 of the preceding paper, and 6, 7, and 8 of this, that

$$1.\quad A = \frac{1}{\pi (a+b)^{\frac{3}{2}}} \times \begin{cases} \frac{2}{cc} + e + fcc + gc^4 \\ \quad + \alpha + \frac{3}{8}\alpha cc + \frac{3.5}{8.8}\alpha c^4. \end{cases}$$

$$2.\quad A' = \frac{1}{\pi (a+b)^{\frac{5}{2}}} \times \begin{cases} \frac{4}{3c^4} + \frac{1}{cc} + h + icc + kc^4 \\ \quad + \frac{3}{4}\alpha + \frac{3.5}{4.12}\alpha cc + \frac{3.5.21}{4.12.32}\alpha c^4. \end{cases}$$

$$3.\quad B = \frac{2a}{b}A - \frac{2}{\pi b (a+b)^{\frac{1}{2}}} \times \begin{cases} \quad p + lcc + mc^4 \\ + 2\alpha + \frac{1}{2}\alpha cc + \frac{9}{2.16}\alpha c^4. \end{cases}$$

$$4.\quad B' = \frac{2}{b}(A'a - A).$$

In which equations, the values of the coefficients are as follows:

$e = 0{\cdot}1931472$,	$h = 0{\cdot}0823604$,	$p = 1{\cdot}3862944$,
$f = 0{\cdot}1036802$,	$i = 0{\cdot}0551502$,	$l = 0{\cdot}3465736$,
$g = 0{\cdot}0687064$,	$k = 0{\cdot}0408309$,	$m = 0{\cdot}1793226$.

13. The constant numbers which will be wanted, in computing the arithmetical values of A and B, are those denoted by e, h, and p, which are given in the preceding Article; and the constant logarithms are the following, which are respectively set down to as many places of figures as are requisite.

L. 2 = 0·3010,300,	L. $\frac{2}{3} = \bar{1}$·8239,087,	L. $\frac{21}{32} = \bar{1}$·817,
L. $\frac{3}{8} = \bar{1}$·5740,3	L. $\frac{3}{4} = \bar{1}$·8750,6,	L. $\frac{1}{2} = \bar{1}$·6989,7,
L. $\frac{5}{8} = \bar{1}$·796,	L. $\frac{5}{12} = \bar{1}$·6197,9,	L. $\frac{9}{16} = \bar{1}$·750
L. $f = \bar{1}$·0157,0	L. $i = \bar{2}$·7415,5	L. $l = \bar{1}$.5398,0
L. $\frac{g}{f} = \bar{1}$·821	L. $\frac{k}{i} = \bar{1}$·869	L. $\frac{m}{l} = \bar{1}$·714
	L. π = 0·4971,499*.	

14. An example, to illustrate the method of computing by these theorems, may now be proper.

Let it be required to compute A and B by the 1st and 3d theorems, in Art. 12. when the two planets are Venus and the Earth.

This arithmetical work may stand as follows, in three columns, the logarithms being in the middle, and the numbers corresponding to them on the two sides; where a distinction is made, which is too obvious to need any description. By this arrangement, a frequent repetition of words, number, and logarithm, will be avoided.

* All these constant logarithms are to be written on a slip of paper, for the sake of expedition in the use of them.

First, for the value of A.

Numbers.		Logarithms.		Numbers.
Here $a = 1{\cdot}5236{,}71$ *		$0{\cdot}1828{,}913$		
and $b = 1{\cdot}4451{,}60$	Ar. c	$\bar{1}{\cdot}8400{,}841$		
$a - b = 0{\cdot}0785{,}11$		$\bar{2}{\cdot}8949{,}305$		
$a + b = 2.9688{,}31$		$0{\cdot}4725{,}855$		
$\frac{a-b}{a+b} = cc$	- -	$\bar{2}{\cdot}4223{,}450$		
$\frac{2}{cc}$		$1{\cdot}8786{,}850$	-	$75{\cdot}62841 = \frac{2}{cc}$
				$0{\cdot}19315 = e$
fcc		$\bar{3}{\cdot}4380{,}4$	-	$0{\cdot}00274 = fcc$
$\frac{f}{g}cc$		$\bar{2}{\cdot}244$		
Sum of these two logs.		$\bar{5}{\cdot}682$	- -	$0{\cdot}00005 = gc^4$
The sum of these four terms is			- -	$75{\cdot}82435$

Having now found the value of the four terms $\frac{2}{cc} + e + fcc + gc^4$, we must next find the value of the three logarithmic terms $\alpha + \frac{3}{8}\alpha cc + \frac{3.5}{8.8}\alpha c^4$, which may quickly be done as follows:

Half the logarithm of cc is $\bar{1}{\cdot}2111{,}725$ = the sine of $9^\circ\ 21'\ 32''{\cdot}28$; and half this angle is $4^\circ\ 40'\ 46''{\cdot}14$, the logarithmic co-tangent of which is $1{\cdot}0869{,}576$; and this common logarithm

* I was favoured with these numbers by Dr. MASKELYNE.

reduced to an hyperbolic logarithm, by Table XXXVII. of DODSON's Calculator, gives - - $2\cdot50281 = \alpha$

α -	$0\cdot39843$	
$\frac{3}{8}cc$ - -	$\bar{3}\cdot99638$	
Sum of these two logs.	$\bar{2}\cdot39481$	- - $0\cdot02482 = \frac{3}{8}\alpha cc$
$\frac{5}{8}cc$ - -	$\bar{2}\cdot218$	
	$\bar{4}\cdot613$	- - $0\cdot00041 = \frac{3.5}{8.8}\alpha c^4$;

The sum of these three terms is - $2\cdot52804$; to which, add the sum of the four terms above found, $75\cdot82435$, and we have

	$1\cdot8940,523$	- $78\cdot35239$ = all the terms.
$\pi(a+b)^{\frac{3}{2}}$	$1\cdot2060,283$	
The difference of these two logarithms is	$0\cdot6880,240$	- $4\cdot87555 = A$.

The value of A being now found, the computation of the value of B will be very easy, since, of the six terms wanted, two are already computed, and the logarithms of all the rest are at hand. This operation may stand as follows, the logarithms being still in the middle.

		$1{\cdot}38630 = \varrho$
	$\bar{3}{\cdot}9621$	$0{\cdot}00916 = lc\varrho$
$\frac{l}{m}cc$	$\bar{2}{\cdot}136$	
Sum of these two logs.	$\bar{4}{\cdot}098$ -	$0{\cdot}00013 = mc^4$
		$5{\cdot}00562 = 2\alpha$
	$\bar{2}{\cdot}51975$	$0{\cdot}03309 = \frac{1}{2}\alpha cc$
$\frac{9}{16}cc$	$\bar{2}{\cdot}172$	
Sum of these two logs.	$\bar{4}{\cdot}692$	$0{\cdot}00049 = \frac{1.9}{2.16}\alpha c^4$
	$0{\cdot}8085{,}343$	$6{\cdot}43479 = \left\{ \text{Sum of these six terms.} \right.$
$\pi(a+b)^{\frac{1}{2}}$	$0{\cdot}7334{,}427$	
Diff. of these two logs.	$0{\cdot}0750{,}916$	$1{\cdot}18875 = A_{,}$
A -	$0.6880{,}240$	
a -	$0.1828{,}913$	
Sum of these two logs.	$0.8709{,}153$ -	$7{\cdot}42874 = Aa$
	$0.7951{,}840$ -	$6{\cdot}23999 = Aa - A_{,}$
$\frac{2}{b}$	$0{\cdot}1411{,}141$	
$\frac{2}{b}(Aa - A_{,})$	$0{\cdot}9362{,}981$	$8{\cdot}63571 = B.$

We have now the values sought, *viz.* $A = 4{\cdot}8756$, and $B = 8{\cdot}6357$; and from these may the values of C, D, E, &c. be easily found, by the equations given in Art. 20. of the preceding paper. I have set them down here only to five places of figures, it being evident, from the value of $a - b$, that the result cannot be depended on to more places than five, which, however, are very sufficient for the purpose.

I have taken notice above, in the computation of B, that, of six terms wanted, two were ready, one of them being a constant quantity, and the other computed in the preceding operation; and it may be remarked, that, if the two terms in which c^4 enters had been omitted, the difference in the result would not have been so much as 2 in the fourth place of decimals, which is inconsiderable. Therefore, of the six terms in this expression, two are given, and two more only need be computed in this case.

And it may be further remarked, that the term e is always given in the expression of the value of A, and that the two terms in which c^4 enters may be omitted, as that will occasion a difference in the result, of only 3 in the fifth place of decimals, which is quite inconsiderable. Of the seven terms, therefore, in Theorem 1. Art. 12. one is given, and four more only need be computed, when Venus and the Earth are the two planets of which the perturbations are to be computed.

15. My avocations calling me off from these delightful speculations, I must now put an end to this paper, without mentioning some other observations which I have made on this subject.

May 6, 1797.